HYPERGRAPHS

Combinatorics of Finite Sets
Hypergraphs
Combinatorics of Finite Sets

Claude BERGE
Centre de Mathematique Sociale
Paris, France

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FOREWORD

For the past forty years, Graph Theory has proved to be an extremely useful tool for solving combinatorial problems, in areas as diverse as Geometry, Algebra, Number Theory, Topology, Operations Research and Optimization. It was thus natural to try and generalise the concept of a graph, in order to attack additional combinatorial problems.

The idea of looking at a family of sets from this standpoint took shape around 1960. In regarding each set as a "generalised edge" and in calling the family itself a "hypergraph", the initial idea was to try to extend certain classical results of Graph Theory such as the theorems of Turán and König. Next, it was noticed that this generalisation often led to simplification; moreover, one single statement, sometimes remarkably simple, could unify several theorems on graphs. It is with this motivation that we have tried in this book to present what has seemed to us to be the most significant work on hypergraphs.

In addition, the theory of hypergraphs is seen to be a very useful tool for the solution of integer optimization problems when the matrix has certain special properties. Thus the reader will come across scheduling problems (Chapter 4), location problems (Chapter 5), etc., which when formulated in terms of hypergraphs, lead to general algorithms. In this way specialists in operations research and mathematical programming have also been kept in mind by emphasizing the applications of the theory.

For pure mathematicians, we have also included several general results on set systems which do not arise from Graph Theory; graphical concepts nevertheless provide an elegant framework for such results, which become easier to visualize.

For students in pure or applied mathematics, we have thought it worthwhile to add at the end of each chapter a collection of related problems. Some are still open but many are straightforward applications of the theory to combinatorial designs, directed graphs, matroids, etc., such consequences being too numerous to include in the text itself.

We wish especially to thank Michel Las Vergnas, and also Dominique de Werra and Dominique de Caen, for their help in the presentation. We express our thanks also to the New York University for permission to include certain chapters of this book which were taught in New York during 1985.

Claude Berge

Note: The longest proofs, and those which are particularly difficult, are indicated in the text by an asterisk; they can easily be skipped on first reading.
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List of standard symbols

\[ R \] \quad \text{Set of real numbers}
\[ N \] \quad \text{Set of integers } \geq 0
\[ Z \] \quad \text{Set of all integers}
\[ \emptyset \] \quad \text{The empty set}
\[ |A| \] \quad \text{Cardinality of the set } A
\[ \{x \mid x \text{ such that } \cdots \} \] \quad \text{Set of } x \text{ such that } ...
\[ (\forall x) \] \quad \text{For every } x
\[ (\exists x) \] \quad \text{There is an } x
\[ a \in A \] \quad \text{a is an element of the set } A
\[ a \notin A \] \quad \text{a is not an element of the set } A
\[ A \cup B \] \quad \text{Union of } A \text{ and } B
\[ A \cap B \] \quad \text{Intersection of } A \text{ and } B
\[ A - B \] \quad \text{A minus } B \text{ (elements of } A \text{ not belonging to } B)\)
\[ A \subset B \] \quad \text{The set } A \text{ is a subset of set } B
\[ A \not\subset B \] \quad A \text{ is not contained in } B
\[ A \times B \] \quad \text{Cartesian product of } A \text{ by } B
\[ (1) \Rightarrow (2) \] \quad \text{Property (1) implies property (2)}
\[ \binom{p}{q} = \frac{p!}{q!(p-q)!} \] \quad \text{Binomial coefficient "} p \text{ choose } q \text{"}
\[ p \equiv q \pmod{k} \] \quad \text{The integer } p \text{ is congruent to } q \text{ modulo } k
\[ \lfloor \frac{p}{q} \rfloor \] \quad \text{Integral part of } \frac{p}{q} \text{ (largest integer } \leq \frac{p}{q})
\[ \lceil \frac{p}{q} \rceil \] \quad \text{Smallest integer } \geq \frac{p}{q}
\[ ((a_{ij})) \] \quad \text{Matrix in which the element in the } i\text{th row and } j\text{th column is } a_{ij}
\[ \det((a_{ij})) \] \quad \text{Determinant}
\[ \log p \] \quad \text{Neperian (natural) logarithm}

For the notations specific to graphs, see the reference: Graphs (C. Berge, Graphs, North Holland, 1985).
Chapter 1

General Concepts

1. Dual Hypergraphs

Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a finite set. A hypergraph on \( X \) is a family \( H = (E_1, E_2, \ldots, E_m) \) of subsets of \( X \) such that

1. \( E_i \neq \emptyset \) \( (i = 1, 2, \ldots, m) \)

2. \( \bigcup_{i=1}^{m} E_i = X \)

A simple hypergraph (or "Sperner family") is a hypergraph \( H = (E_1, E_2, \ldots, E_m) \) such that

3. \( E_i \subseteq E_j \Rightarrow i = j \)

The elements \( x_1, x_2, \ldots, x_n \) of \( X \) are called vertices, and the sets \( E_1, E_2, \ldots, E_m \) are the edges of the hypergraph. A simple graph is a simple hypergraph each of whose edges has cardinality 2; a multigraph (with loops and multiple edges) is a hypergraph in which each edge has cardinality \( \leq 2 \). Nonetheless we shall not consider isolated points of a graph to be vertices.

A hypergraph \( H \) may be drawn as a set of points representing the vertices. The edge \( E_j \) is represented by a continuous curve joining the two elements if \( |E_j| = 2 \), by a loop if \( |E_j| = 1 \), and by a simple closed curve enclosing the elements if \( |E_j| \geq 3 \).

One may also define a hypergraph by its incidence matrix \( A = (a_{ij}) \), with columns representing the edges \( E_1, E_2, \ldots, E_m \) and rows representing the vertices \( x_1, x_2, \ldots, x_n \), where \( a_{ij} = 0 \) if \( x_i \notin E_j \), \( a_{ij} = 1 \) if \( x_i \in E_j \) (cf. Figure 1).
The dual of a hypergraph \( H = (E_1, E_2, \ldots, E_m) \) on \( X \) is a hypergraph \( H^* = (X_1, X_2, \ldots, X_n) \) whose vertices \( e_1, e_2, \ldots, e_m \) correspond to the edges of \( H \), and with edges

\[
X_i = \{ e_j : x_i \in E_j \; \text{in} \; H \}
\]

\( H^* \) clearly satisfies both conditions (1) and (2).

It is easily seen that the incidence matrix of \( H^* \) is the transpose of the incidence matrix of \( H \) and so we have \((H^*)^* = H\).

Figure 2. The dual hypergraph of the hypergraph in Figure 1.

As for a graph, the order of \( H \), denoted by \( n(H) \), is the number of vertices. The number of edges will be denoted by \( m(H) \). Further the rank is \( r(H) = \max_j |E_j| \), the anti-rank is \( s(H) = \min_j |E_j| \); if \( r(H) = s(H) \) we say that \( H \) is a uniform hypergraph;
a simple uniform hypergraph of rank $r$ will also be called $r$-uniform, and in this case it will be understood that there is no repeated edge.

For a set $J \subseteq \{1,2,\ldots,m\}$ we call the family 
\[ H' = (E_j/ j \in J) \]
the partial hypergraph generated by the set $J$. The set of vertices of $H'$ is a nonempty subset of $X$.

For a set $A \subseteq X$ we call the family 
\[ H_A = (E_j \cap A/ 1 \leq j \leq m, E_j \cap A \neq \emptyset) \]
the sub-hypergraph induced by the set $A$. (We define partial sub-hypergraphs etc. in a similar fashion).

**Proposition.** The dual of a subhypergraph of $H$ is a partial hypergraph of the dual hypergraph $H^*$.

In the case of hypergraphs of rank 2 these reduce to the familiar definitions for graphs. All the concepts of graph theory may thus be generalised to hypergraphs which will allow us to find stronger theorems, and applications to objects other than graphs. Further the formulation of a combinatorial problem in terms of hypergraphs sometimes has the advantage of providing a remarkably simple statement having a familiar form.

A stronger result may be much easier to prove than the weak result!

2. Degrees

The other definitions from graph theory which may be extended without ambiguity to a hypergraph $H$ are the following:

For $x \in X$, define the star $H(x)$ with centre $x$ to be the partial hypergraph formed by the edges containing $x$. Define the degree $d_H(x)$ of $x$ to be the number of edges of $H(x)$, so $d_H(x) = m(H(x))$.

The maximum degree of the hypergraph $H$ will always be denoted by 
\[ \Delta(H) = \max_{x \in X} d_H(x). \]
A hypergraph in which all vertices have the same degree is said to be regular.
Note that $\Delta(H) = r(H^*)$, and that the dual of a regular hypergraph is uniform.

For a hypergraph $H$ of order $n$, the degrees $d_H(x_i) = d_i$ in decreasing order form an $n$-tuple $d_1 \geq d_2 \geq \cdots \geq d_n$ whose properties can be characterised if $H$ is a simple graph (Erdős, Gallai [1960], cf. Graphs, Ch. 6, Th. 6). In general

**Proposition 1.** An $n$-tuple $d_1 \geq d_2 \geq \cdots \geq d_n$ is the degree sequence of a uniform hypergraph of rank $r$ and order $n$ (possibly with repeated edges) if and only if $
sum_{i=1}^{n} d_i$ is a multiple of $r$ and $d_n \geq 1$.

**Proof.** Given such an $n$-tuple $d_1 \geq d_2 \geq \cdots \geq d_n$, we wish to construct the edges of a hypergraph $H$ one by one on the set $\{x_1, x_2, \ldots, x_n\}$.

In the first step, associate with each vertex $x_i$ a weight $d^1_i = d_i$ and form the first edge $E_1$ by taking the $r$ vertices of greatest weight. In the second step, associate with vertex $x_i$ the weight

$$d^2_i = \begin{cases} 
    d^1_i & \text{if } x_i \notin E_1 \\
    d^1_i - 1 & \text{if } x_i \in E_1
\end{cases}$$

Form $E_2$ by taking the $r$ vertices of greatest weight, etc. If $\sum d_i = mr$ we obtain $H$ with the edges $E_1, E_2, \ldots, E_m$, and $d_H(x_i) = d_i$ for $i = 1, 2, \ldots, n$.

A hypergraph is connected if the intersection graph of the edges is connected. Then we have

**Proposition 2** (Tusyadej [1978]). An $n$-tuple $d_1 \geq d_2 \geq \cdots \geq d_n$ is the degree sequence of a connected uniform hypergraph of rank $r$ if and only if

1. $\sum_{i=1}^{n} d_i$ is a multiple of $r$,
2. $d_i \geq 1$ (i = 1, 2, n),
3. $\sum_{i=1}^{n} d_i \geq \frac{r(n-1)}{r-1}$.
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(4) \[ d_1 \leq m = \frac{\Sigma d_i}{r}. \]

(For extensions to non-uniform hypergraphs, cf. Boonyasombat [1984]).

**Theorem 1** (Gale [1957], Ryser [1957]). Given \( m \) integers \( r_1, r_2, \ldots, r_m \) and an \( n \)-tuple of integers \( d_1 \geq d_2 \geq \cdots \geq d_n \), there exists a hypergraph \( H = (E_1, E_2, \ldots, E_m) \) on a set \( X = \{x_1, x_2, \ldots, x_n\} \) such that \( d_H(x_i) = d_i \) for \( i \leq n \) and \( |E_j| = r_j \) for \( j \leq m \) if and only if

\[(1) \quad \sum_{j=1}^{m} \min\{r_j,k\} \geq d_1 + d_2 + \cdots + d_k \quad (k < n)\]

\[(2) \quad \sum_{j=1}^{m} r_j = d_1 + d_2 + \cdots + d_n.\]

**Proof.** We deduce this immediately from the theory of network flows (corollary to theorem 3, Ch.5 in *Graphs*). Indeed, construct a network flow with vertices the points \( j = 1, 2, \ldots, m \) and \( x_1, x_2, \ldots, x_n \), with a source \( a \) and a sink \( z \). The arcs are

- all arcs \((a,j)\) with capacity \( r_j \)
- all arcs \((x_i,z)\) with capacity \( d_i \)
- all arcs \((j,x_i)\) with capacity 1.

It suffices to show that there exists an integer flow satisfying the capacities, saturating each of the arcs \((j,z)\) entering the sink \( z \), that is to say that the maximum flow which can enter set \( \{x_i/i \in I\} \) is always greater than or equal to the sum \( \sum_{i \in I} d_i \), for all \( I \subseteq \{1,2,\ldots,n\} \). (Further, we note that thanks to the network flow theorem we may always suppose that such a flow never leaves an "entry" arc or an "exit" arc.)

**Open Problem.** Find a necessary and sufficient condition for an \( m \)-tuple \((r_j)\) and an \( n \)-tuple \((d_i)\) to be respectively the \(|E_j|\) and the \( d_H(x_i) \) of a simple hypergraph \( H \).

Let \( r, n \) be integers, \( 1 \leq r \leq n \). We define the \( r \)-uniform complete hypergraph of order \( n \) (or the \( r \)-complete hypergraph) to be a hypergraph denoted \( K^r_n \) consisting of all the \( r \)-subsets of a set \( X \) of cardinality \( n \). We may now state in a complete form the celebrated *Sperner's theorem* [1928]; in fact the inequality (1), which allows for a
simple proof was discovered (independently) much later by Yamamoto, Meshalkin, Lubell and Bollobás.

**Theorem 2** (Sperner [1928]; proof by Yamamoto, Meshalkin, Lubell, Bollobás). *Every simple hypergraph* $H$ *of order* $n$ *satisfies*

\[ \sum_{E \in H} \left( \frac{n}{|E|} \right)^{-1} \leq 1. \]

Further, the number of edges $m(H)$ satisfies

\[ m(H) \leq \left( \frac{n}{\lfloor n/2 \rfloor} \right) \]

For $n = 2h$ even, equality in (2) is attained if and only if $H$ is the hypergraph $K^h_n$. For $n = 2h-1$ odd, equality in (2) is attained if and only if $H$ is the hypergraph $K^h_n$ or the hypergraph $K^{h+1}_n$.

**Proof.** Let $X$ be a finite set of cardinality $n$. Consider a directed graph $G$ with vertices the subsets of $X$, and with an arc from $A \subset X$ to $B \subset X$ if $A \subset B$ and $|A| = |B| - 1$.

Let $E \in H$, the number of paths in the graph $G$ from the vertex $\emptyset$ to the vertex $E$ is $|E|!$, thus the total number of paths from $\emptyset$ to $X$ is $n! \geq \sum_{E \in H} (|E|!)(n-|E|)!$ (as $H$ is a simple hypergraph, a path passing through $E$ cannot pass through $E' \in H$, $E' \neq E$). We thus deduce inequality (1).

For the second part,

\[ \left( \frac{n}{|E|} \right) \leq \left( \frac{n}{\lfloor n/2 \rfloor} \right). \]

whence

\[ 1 \geq \sum_{E \in H} \left( \frac{n}{|E|} \right)^{-1} \geq m(H) \left( \frac{n}{\lfloor n/2 \rfloor} \right)^{-1}. \]

We immediately deduce inequality (2).

Let $H$ be a hypergraph satisfying equality in (2). Then for all $E \in H$,
(3) \[ \binom{n}{|E|} = \binom{n}{[n/2]} . \]

If \( n = 2h \) is even, (3) implies that \( H \) is \( h \)-uniform, and since \( m(H) = \binom{n}{h} \) we have \( H = K^h_n \), and the proof is achieved.

If \( n = 2h+1 \), (3) implies that \( h \leq |E| \leq h+1 \) for all \( E \subseteq H \). Let \( X^h \) be the set of vertices in \( G \) which represent edges of \( H \) with cardinality \( k \); the set \( X^h \cup X^h_{h+1} \) is a stable set of \( G \), and \( m(H) = |X^h \cup X^h_{h+1}| \).

The number of arcs of \( G \) leaving \( X^h \) is equal to \( |X^h| (n-h) \); the number of arcs entering the image \( \Gamma X^h \) of \( X^h \) is \( |\Gamma X^h| (h+1) \). Thus

\[ |\Gamma X^h| (h+1) \geq |X^h| (n-h), \]

or

\[ |\Gamma X^h| \geq \frac{2h+1-h}{h+1} |X^h| = |X^h|. \]

If \( X^h \) is non-empty and is not the set \( P_h(X) \) of all \( h \)-subsets of \( X \), the above inequality is strict (because the bipartite subgraph of \( G \) generated by the \( h \)-subsets and \((h+1)\)-subsets is connected), whence

\[ m(H) = |X^h| + |X^h_{h+1}| \leq |X^h| + |P_{h+1}(X) - \Gamma X^h| . \]

\[ < |X^h| + \binom{n}{h+1} - |X^h| = \binom{n}{h+1} . \]

Thus, equality in (2) is possible only if \( X^h = \emptyset \) or \( X^h = P_h(X) \), i.e. if \( H = K^h_n \) or \( K^{h+1}_n \).

Q.E.D.


To generalise graphs without "pendent" vertices, we consider the following class of hypergraphs; a hypergraph \( H \) is said to be separable if for every vertex \( x \), the intersection of the edges containing \( x \) is the singleton \( \{x\} \) i.e. if \( \bigcap_{E \in H(x)} E = \{x\} \).

**Corollary.** If an \( n \)-tuple \( d_1 \geq d_2 \geq \cdots \geq d_n \) of positive integers is the degree sequence of a separable hypergraph \( H = (E_1, \ldots, E_m) \) then
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Essentially \( H \) is separable if and only if its dual \( H^* \) is a simple hypergraph, which implies, by Theorem 2,

\[
\sum_{i=1}^{n} \left( \frac{m}{|X_i|} \right)^{-1} \leq 1.
\]

Q.E.D.

To generalise simple graphs, we say that a hypergraph \( H = (E_1, E_2, \ldots, E_m) \) is linear if \( |E_i \cap E_j| \leq 1 \) for \( i \neq j \). For example, the hypergraphs of Figures 1, 2 are linear.

We have immediately

Proposition 3. The dual of a linear hypergraph is also linear.

Indeed, if \( H \) is linear, two edges \( X_i \) and \( X_j \) in \( H^* \) cannot intersect in two distinct points \( e_1, e_2 \), as then, in \( H \), \( E_1 \supset \{x_1, x_2\} \), \( E_2 \supset \{x_1, x_2\} \), contradicting \( |E_1 \cap E_2| \leq 1 \).

Theorem 3. For every linear hypergraph \( H \) of order \( n \), we have

\[
(1) \quad \sum_{E \in H} \left( \frac{|E|}{2} \right) \leq \binom{n}{2}.
\]

If in addition, \( H \) is \( r \)-uniform, then the number of edges satisfies

\[
(2) \quad m(H) \leq \frac{n(n-1)}{r(r-1)}.
\]

The bound in (2) is attained if and only if \( H \) is a Steiner system \( S(2,r,n) \).

For, the number of pairs \( x,y \) which are contained in a same edge of \( H \) is

\[
\sum_{E \in H} \left( \frac{|E|}{2} \right) \leq \binom{n}{2}
\]

whence we have (1). If \( H \) is \( r \)-uniform, (2) follows.

A Steiner system \( S(2,r,n) \) is an \( r \)-uniform hypergraph on \( X \), with \( |X| = n \), in which every pair of vertices is contained in exactly one edge. A necessary and sufficient condition for the existence of an \( S(2,3,n) \) system, due to T.P. Kirkman [1847], is
that \( n \equiv 1 \) or \( 3 \) (mod 6).

To exclude some values of \( r \) it is easily seen that the following are necessary conditions for the existence of \( S(2,r,n) \) systems:

1. \( \binom{n}{2}^{-1} \) is an integer;

2. \( (n-1)(r-1)^{-1} \) is an integer.

These conditions are necessary and sufficient for \( r = 3,4 \) (Hanani). For \( r = 6 \) these conditions are sufficient with a single exception: no \( S(2,6,21) \) system exists. Wilson [1972] has further shown that if \( r \) is a prime power and if \( n \) is sufficiently large then (1) and (2) are necessary and sufficient.

For all questions on existence and enumeration of \( S(2,r,n) \) systems, see Lindner and Rosa [1980]. We give here a list of \( S(2,r,n) \) systems known for small values of \( r \) and of \( n \):

\[
\begin{align*}
S(2,3,7) & \quad \text{De Pasquale [1899], Brunel [1901], Cole [1913]} \\
S(2,3,9) & \quad \text{De Pasquale [1899], Brunel [1901], Cole [1913]} \\
S(2,4,13) & \quad \text{De Pasquale [1899], Brunel [1901], Cole [1913]} \\
S(2,3,15) & \quad \text{Cole [1917], White [1919], Fischer [1940]} \\
S(2,4,16) & \quad \text{Witt [1938]} \\
S(2,3,19) & \quad \text{Deherder [1976]} \\
S(2,3,21) & \quad \text{Wilson [1974]} \\
S(2,5,21) & \quad \text{Witt [1938]} \\
S(2,3,25) & \quad \text{Wilson [1974]} \\
S(2,4,25) & \quad \text{Brouwer, Rokowska [1977]} \\
S(2,5,25) & \quad \text{McInnes [1977]} \\
S(2,3,27) & \quad \text{McInnes [1977]} \\
S(2,4,28) & \quad \text{Rokowska [1977]} \\
\end{align*}
\]

We deduce that the bound in (2) of Theorem 3 is the best possible for \( n = 7, r = 3 \); or for \( n = 9, r = 3 \); etc.