Graphs of Groups on Surfaces
Interactions and Models

Arthur T. WHITE
GRAPHS OF GROUPS ON SURFACES
INTERACTIONS AND MODELS
NORTH-HOLLAND MATHEMATICS STUDIES 188
(Continuation of the Notas de Matemática)

Editor: Saul LUBKIN
University of Rochester
New York, U.S.A.

2001
ELSEVIER
Amsterdam - London - New York - Oxford - Paris - Shannon - Tokyo
Foreword

Topological graph theory began in the middle of the eighteenth century, with Euler’s polyhedral identity; it took heightened interest in the latter part of the nineteenth century, with Heawood’s Map-Color Conjecture; and it emerged as a field of study in its own right with the Complete Graph Theorem of Ringel and Youngs in 1968 (completing the proof that Heawood had started). Ringel’s books *Färbungsprobleme auf Flächen und Graphen* (VEB Deutscher Verlag der Wissenschaften, Berlin, 1959) and *Map Color Theorem* (Springer-Verlag, Berlin, 1974) are, as the titles suggest, devoted to the interplay between the conjecture and the theorem mentioned above. The first book devoted to topological graph theory as an independent field of study is my *Graphs, Groups and Surfaces* (North-Holland, Amsterdam, 1973; Revised Edition, 1984). Related books include Gross and Tucker’s *Topological Graph Theory* (Wiley Interscience, New York, 1987), Bonnington and Little’s *The Foundations of Topological Graph Theory* (Springer, New York, 1995), and Mohar and Thomassen’s *Graphs on Surfaces* [MT1]. See also Archdeacon’s “Topological Graph Theory” [A12], a survey article with 271 references.

Whereas the two editions of *Graphs, Groups and Surfaces* introduced topological graph theory in general, with a particular emphasis on various interactions among the three structures of the title (see Figure 0-1) as well as models of hypergraphs, block designs, and compositions of English church-bell music, the present book will also use suitable imbeddings of graphs of groups on surfaces to model finite fields and finite geometries. The material on change ringing is greatly updated, and introductions to enumerative and random topological graph theory have been added. The unifying concept is that of a Cayley map: the lift, as a branched covering space, of an index-one voltage graph imbedding, for a fixed group and generating set. (The latter consists of one vertex, a directed loop for each generator, and a particular imbedding of the loop digraph. The covering graph is then a Cayley color graph.)

I have attempted to make all this material, with its fascinating interconnections, readily accessible to a beginning graduate (or an advanced undergraduate) student (introductory knowledge of both group theory and topology would be helpful), while at the same time providing the research mathematician with a useful reference book in topological graph theory. The latter aspect will not be comprehensive, however, as the field is not too broad to allow this reasonably. The focus will be on beautiful connections, both elementary and deep, within mathematics that can best be described by the intuitively pleasing device of imbedding graphs of groups on surfaces. Several peripheral (but significant) results are stated without proof. An effort has been made to provide
those proofs of theorems which are most indicative of the charm and beauty of the subject and which illustrate the techniques employed. Proofs missing in the text can be supplied by the reader, as part of the problem sets (of the total of 297 problems, 30 have been designated as “difficult” (*) and 9 as “unsolved” (**)), or can be found in the references.

A bibliography is provided, for future reading; items h and n, f and i, k respectively are especially suitable for more extensive treatments of the theories of graphs, of groups, and of surfaces, which are seen interacting in this text.

I thank everyone who read either edition of *Graphs, Groups and Surfaces*, particularly all those who sent me comments and corrections. I especially thank Margo Chapman for preparing the manuscript, and Ramón Figueroa-Centeno for producing the figures and greatly assisting with the preparation, of the present volume. I also thank Jim Laser, Michelle Schultz, Jay Treiman, and Mary Van Popering for their considerable help. Finally I thank Western Michigan University for funding the sabbatical year during which this book was written and the Mathematical Institute and Wolfson College, University of Oxford, for hosting my sabbatical visit.

A.T.W. Kalamazoo January 2001
Figure 0-1.
This page intentionally left blank


## Contents

Chapter 1. **HISTORICAL SETTING**  

Chapter 2. **A BRIEF INTRODUCTION TO GRAPH THEORY**  
  2-1. Definition of a Graph  
  2-2. Variations of Graphs  
  2-3. Additional Definitions  
  2-4. Operations on Graphs  
  2-5. Problems  

Chapter 3. **THE AUTOMORPHISM GROUP OF A GRAPH**  
  3-1. Definitions  
  3-2. Operations on Permutations Groups  
  3-3. Computing Automorphism Groups of Graphs  
  3-4. Graphs with a Given Automorphism Group  
  3-5. Problems  

Chapter 4. **THE CAYLEY COLOR GRAPH OF A GROUP PRESENTATION**  
  4-1. Definitions  
  4-2. Automorphisms  
  4-3. Properties  
  4-4. Products  
  4-5. Cayley Graphs  
  4-6. Problems  

Chapter 5. **AN INTRODUCTION TO SURFACE TOPOLOGY**  
  5-1. Definitions  
  5-2. Surfaces and Other 2-manifolds  
  5-3. The Characteristic of a Surface
Chapter 6. IMBEDDING PROBLEMS IN GRAPH THEORY 49
6-1. Answers to Some Imbedding Questions 49
6-2. Definition of “Imbedding” 52
6-3. The Genus of a Graph 52
6-4. The Maximum Genus of a Graph 55
6-5. Genus Formulae for Graphs 58
6-6. Rotation Schemes 62
6-7. Imbedding Graphs on Pseudosurfaces 64
6-8. Other Topological Parameters for Graphs 66
6-9. Applications 69
6-10. Problems 70

Chapter 7. THE GENUS OF A GROUP 73
7-1. Imbeddings of Cayley Color graphs 73
7-2. Genus Formulae for Groups 77
7-3. Related Results 84
7-4. The Characteristic of a Group 86
7-5. Problems 87

Chapter 8. MAP-COLORING PROBLEMS 89
8-1. Definitions and the Six-Color Theorem 90
8-2. The Five-Color Theorem 90
8-3. The Four-Color Theorem 91
8-4. Other Map-Coloring Problems:
The Heawood Map-Coloring Theorem 92
8-5. A Related Problem 96
8-6. A Four-Color Theorem for the Torus 98
8-7. A Nine-Color Theorem for the Torus and Klein Bottle 101
8-8. k-degenerate Graphs 101
8-9. Coloring Graphs on Pseudosurfaces 103
8-10. The Cochromatic Number of Surfaces 105
<table>
<thead>
<tr>
<th>Chapter 9</th>
<th>QUOTIENT GRAPHS AND QUOTIENT MANIFOLDS: CURRENT GRAPHS AND THE COMPLETE GRAPH THEOREM</th>
<th>107</th>
</tr>
</thead>
<tbody>
<tr>
<td>9-1</td>
<td>The Genus of $K_n$</td>
<td>107</td>
</tr>
<tr>
<td>9-2</td>
<td>The Theory of Current Graphs as Applied to $K_n$</td>
<td>109</td>
</tr>
<tr>
<td>9-3</td>
<td>A Hint of Things to Come</td>
<td>114</td>
</tr>
<tr>
<td>9-4</td>
<td>Problems</td>
<td>116</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Chapter 10</th>
<th>VOLTAGE GRAPHS</th>
<th>119</th>
</tr>
</thead>
<tbody>
<tr>
<td>10-1</td>
<td>Covering Spaces</td>
<td>119</td>
</tr>
<tr>
<td>10-2</td>
<td>Voltage Graphs</td>
<td>121</td>
</tr>
<tr>
<td>10-3</td>
<td>Examples</td>
<td>128</td>
</tr>
<tr>
<td>10-4</td>
<td>The Heawood Map-coloring Theorem (again)</td>
<td>134</td>
</tr>
<tr>
<td>10-5</td>
<td>Strong Tensor Products</td>
<td>135</td>
</tr>
<tr>
<td>10-6</td>
<td>Covering Graphs and Graphical Products</td>
<td>136</td>
</tr>
<tr>
<td>10-7</td>
<td>Problems</td>
<td>139</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Chapter 11</th>
<th>NONORIENTABLE GRAPH IMBEDDINGS</th>
<th>143</th>
</tr>
</thead>
<tbody>
<tr>
<td>11-1</td>
<td>General Theory</td>
<td>143</td>
</tr>
<tr>
<td>11-2</td>
<td>Nonorientable Covering Spaces</td>
<td>145</td>
</tr>
<tr>
<td>11-3</td>
<td>Nonorientable Voltage Graph Imbeddings</td>
<td>146</td>
</tr>
<tr>
<td>11-4</td>
<td>Examples</td>
<td>147</td>
</tr>
<tr>
<td>11-5</td>
<td>The Heawood Map-coloring Theorem; Nonorientable Version</td>
<td>151</td>
</tr>
<tr>
<td>11-6</td>
<td>Other Results</td>
<td>152</td>
</tr>
<tr>
<td>11-7</td>
<td>Problems</td>
<td>154</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Chapter 12</th>
<th>BLOCK DESIGNS</th>
<th>157</th>
</tr>
</thead>
<tbody>
<tr>
<td>12-1</td>
<td>Balanced Incomplete Block Designs</td>
<td>157</td>
</tr>
<tr>
<td>12-2</td>
<td>BIBDs and Graph Imbeddings</td>
<td>158</td>
</tr>
<tr>
<td>12-3</td>
<td>Examples</td>
<td>160</td>
</tr>
<tr>
<td>12-4</td>
<td>Strongly Regular Graphs</td>
<td>161</td>
</tr>
<tr>
<td>12-5</td>
<td>Partially Balanced Incomplete Block Designs</td>
<td>162</td>
</tr>
<tr>
<td>12-6</td>
<td>PBIBDs and Graph Imbeddings</td>
<td>164</td>
</tr>
</tbody>
</table>
12-7. Examples 165
12-8. Doubling a PBIBD 168
12-9. Problems 169

Chapter 13. HYPERGRAPH IMBEDDINGS 173
  13-1. Hypergraphs 173
  13-2. Associated Bipartite Graphs 175
  13-3. Imbedding Theory for Hypergraphs 175
  13-4. The Genus of a Hypergraph 178
  13-5. The Heawood Map-Coloring Theorem, for Hypergraphs 179
  13-6. The Genus of a Block Design 180
  13-7. An Example 181
  13-8. Nonorientable Analogs 183
  13-9. Problems 183

Chapter 14. FINITE FIELDS ON SURFACES 185
  14-1. Graphs Modelling Finite Rings 185
  14-2. Basic Theorems About Finite Fields 187
  14-3. The genus of \( F_p \) 189
  14-4. The Genus of \( F_{p^r} \) 191
  14-5. Further Results 194
  14-6. Problems 196

Chapter 15. FINITE GEOMETRIES ON SURFACES 199
  15-1. Axiom Systems for Geometries 199
  15-2. \( n \)-Point Geometry 200
  15-3. The Geometries of Fano, Pappus, and Desargues 201
  15-4. Block Designs as Models for Geometries 205
  15-5. Surface Models for Geometries 206
  15-6. Fano, Pappus, and Desargues Revisited 207
  15-7. 3-Configurations 209
  15-10. Ten Models for \( AG(2, 3) \) 228
  15-11. Completing the Euclidean Plane 231
CONTENTS

15-12. Problems 232

Chapter 16. MAP AUTOMORPHISM GROUPS 235
16-1. Map Automorphisms 235
16-2. Symmetrical Maps 240
16-3. Cayley Maps 243
16-4. Complete Maps 247
16-5. Other Symmetrical Maps 249
16-6. Self-Complementary Graphs 250
16-7. Self-dual Maps 251
16-8. Paley Maps 255
16-9. Problems 264

Chapter 17. ENUMERATING GRAPH IMBEDDINGS 267
17-1. Counting Labelled Orientable 2-Cell Imbeddings 267
17-2. Counting Unlabelled Orientable 2-Cell Imbeddings 274
17-3. The Average Number of Symmetries 276
17-4. Problems 278

Chapter 18. RANDOM TOPOLOGICAL GRAPH THEORY 281
18-1. Model I 282
18-2. Model II 285
18-3. Model III 287
18-4. Model IV 288
18-5. Model V 289
18-6. Model VI: Random Cayley Maps 290
18-7. Problems 293

Chapter 19. CHANGE RINGING 295
19-1. The Setting 295
19-2. A Mathematical Model 299
19-3. Minimus 301
19-4. Doubles 305
19-5. Minor 311
19-6. Triples and Fabian Stedman 312
CHAPTER 1

HISTORICAL SETTING

In coloring the regions of a map, one must take care to color differently any two countries sharing a common boundary line, so that the two countries can be distinguished. One would think that an economyminded map-maker would wish to minimize the number of colors to be used for a given map, although there appears to be no historical evidence of any such effort. Nevertheless a conjecture was made, about one and one half centuries ago, to the effect that four colors would always suffice for a map drawn on the sphere, the regions of which were all connected. The first reported mention of this problem (see [BCL1] and [O1]) was by Francis Guthrie, through his brother Frederick and Augustus de Morgan, in 1852. The first written references were by Cayley, in 1878 and 1879. Incorrect "proofs" of the Four Color Conjecture were published soon after by Kempe and Tait. The error in Kempe's "proof" was found by Heawood [H4] in 1890; this error has reappeared in various guises in subsequent years. Ore and Stemple [OS1] showed that any counterexample to the conjecture must involve a map of at least 72 regions. The conjecture continued to provide one of the most famous unsolved problems in mathematics, until Appel and Haken affirmed it in 1976. [AH1].

It is an astonishing fact that several related, seemingly much more difficult, map-coloring problems were completely solved prior to the four color problem. Chief among these is the Heawood Map-coloring Conjecture, which gives the chromatic number for every closed 2-manifold other than the sphere; we state the orientable case:

$$\chi(S_k) = f(k) = \left[\frac{7 + \sqrt{1 + 48k}}{2}\right], \text{ for } k > 0,$$

where $k$ is the genus of the closed orientable 2-manifold $S_k$. Heawood showed in 1870 [H4] that $\chi(S_k) \leq f(k)$, and in 1891 Heffter [H5] showed the reverse inequality for a possibly infinite set of natural numbers $k$; almost eight decades passed before it was shown that $\chi(S_k) \geq f(k)$, for all $k > 0$. In 1965 this problem was given the place of honor on the dust jacket for Tietze's Famous Problems of Mathematics [T7]. An outline of the major portion of the solution now follows.

The dual of a map drawn on $S_k$ is a pseudograph imbedded in $S_k$, and it can be shown (see Section 8-4) that $\chi(S_k) \geq f(k)$, for $k > 0,
provided the complete graph $K_n$ has genus given by

\[ (*) \gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil, \quad n \geq 7. \]

Heawood established (*) for $n = 7$ in 1890, and Heffter for $8 \leq n \leq 12$ in 1891; Ringel handled $n = 13$ in 1952. The first major breakthrough occurred in 1954, when Ringel showed (*) for $n \equiv 5 \pmod{12}$. During 1961 - 1965, Ringel treated the residue cases $7, 10,$ and $3 \pmod{12}$, while independently Gustin settled the cases $3, 4,$ and $7$. Gustin's method involved the powerful and beautiful idea of quotient graph and quotient manifold, and relies upon the fact that $K_n$ can be regarded as a Cayley color graph for a group presentation; thus graph theory, group theory, and surface topology combined to solve this famous problem of mathematics.

In 1965, Terry, Welch, and Youngs announced their solution to case 0. Gustin, Ringel, and Youngs finished the remaining residue cases \( \pmod{12} \), except for the isolated values $n = 18, 20,$ and $23$; their work was announced in 1968 [RY1]. In 1969, Jean Mayer (a Professor of French Literature) [M4] eliminated the last three obstinate graphs by \textit{ad hoc} techniques.

Much of the work of Ringel, Terry, Welch, and Youngs was made possible by Gustin's theory of quotient graphs and quotient manifolds; this theory was developed and modified by Youngs, who also introduced the theory of vortices [Y3]. The theory is considerably more general than was needed to prove the Heawood Map-coloring Theorem, and was unified and developed in more generality by Jacques [J3], in 1969. Jacques' results are accessible in Chapter 9 of [W15], together with many applications to other imbedding problems in graph theory. This was a focal point of that text, and it illustrates vividly the fruitful interaction among graphs, groups and surfaces.

We continue this development through the theory of voltage graphs (introduced by Gross [G4] and by Gross and Alpert ([GA1], [GA2])) and by extending to nonorientable imbeddings. We consider the related structures of block designs and hypergraph imbeddings. In map automorphism groups we study groups acting on graphs of groups on surfaces. Cayley maps dominate our considerations. They allow concrete models of finite geometries and finite fields, as well as for finite groups, and we study them also in contexts of enumerative and of topological graph theory. Finally, in studying change ringing, we use graphs of groups on surfaces to compose pieces of music for English church bells.

The conjunctions of graph theory, group theory, and surface topology described above are foreshadowed, in this text, by several pairwise interactions among these three disciplines. The Heawood Map-coloring
Theorem is proved by finding, for each surface, a graph of largest chromatic number that can be drawn on that surface. Equivalently (as it turns out) we find, for each complete graph, the surface of smallest genus in which it can be drawn. The extension of this latter problem to arbitrary graphs is natural; the solution is particularly elegant for graphs which are the Cayley color graphs of a group. We are led in turn to the problem of finding, for a given group, a surface of minimum genus which represents the group in some way.

Dyck [D7] (see also Burnside [B21], Chapters 18 and 19) considered maps, on surfaces, that are transformed into themselves in accordance with the fixed group $\Gamma$, acting transitively on the regions of the map. Any such map gives an upper bound for the parameter $\gamma(\Gamma)$ discussed in Chapter 7 of this text, as a "dual" formed in terms of Burnside's white regions gives a Cayley color graph for $\Gamma$. (Cayley [C4] defined his color graphs as complete symmetric digraphs, corresponding to the choice $\Gamma$ less the identity element as a generating set for $\Gamma$; it is sensible to extend his definition to any generating set for the group in question.) Brahana [B18, B19] studied groups represented by regular maps on surfaces; these maps correspond to presentations on two generators, one of which is of order two. In this context the group acts transitively on the edges of the map, and again an upper bound for $\gamma(\Gamma)$ is obtained. In Chapter 7, we regard $\Gamma$ as acting transitively on the vertices of the map induced by imbedding a Cayley color graph $C_\Delta(\Gamma)$ for $\Gamma$ in a surface; in Chapter 4, we show that the automorphism group of $C_\Delta(\Gamma)$ is isomorphic to $\Gamma$, independent of the generating set $\Delta$ selected for $\Gamma$, so that in this sense $C_\Delta(\Gamma)$ provides a "picture" of $\Gamma$. But more: many properties of $\Gamma$, such as commutivity, normality of certain subgroups, the entire multiplication table, can be "seen" from the picture provided by $C_\Delta(\Gamma)$. Thus it is natural to seek the simplest surface on which to draw this picture; this is given by the parameter $\gamma(\Gamma)$.

This point of view may give a surface of lower genus for a given group than the other two approaches listed above; for example, the group $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_4$ is toroidal for Dyck (or Burnside) and for Brahana, yet $\gamma(\mathbb{Z}_2 \times \mathbb{Z}_4) = 0$.

There is one correspondence depicted in Figure 0-1 which we discuss only briefly in this text: to every surface $S_k$ there corresponds a unique group, $\Omega(S_k)$, called the fundamental group of the surface; the groups $\Omega(S_k)$ have been completely determined – they are given by $2k$ generators $a_1, b_1, \ldots, a_k, b_k$ and the single defining relation

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_k b_k a_k^{-1} b_k^{-1} = e$$

(see, for example, [S11].) Each of the other five correspondences illustrated in Figure 0-1 (where the inner triangle commutes, for proper choice of $\Delta$) is germane, as outlined above, to the conjunction of graph
theory, group theory, and surface topology described in this introduction and which we now begin to develop.
CHAPTER 2

A BRIEF INTRODUCTION TO GRAPH THEORY

In this chapter we introduce basic terminology from the theory of graphs that will be used in this text. We will give several binary operations on graphs; these will enable us to construct more complicated graphs, and hence to build up our store of examples of frequently encountered graphs.

We emphasize that the material introduced here is primarily for the purpose of later use in this text; for a considerably more thorough introduction to graph theory, see [CL1] or [H3].

2-1. Definition of a Graph

Def. 2-1. A graph $G$ consists of a finite non-empty set $V(G)$ of vertices together with a set $E(G)$ of unordered pairs of distinct vertices, called edges. If $x = \{u, v\} \in E(G)$, for $u, v \in V(G)$, we say that $u$ and $v$ are adjacent vertices, and that vertex $u$ and edge $x$ are incident with each other, as are $v$ and $x$. We also say that the edges $\{u, v\}$ and $\{u, w\}, w \neq v$, are adjacent. The degree, $d(v)$, of a vertex $v$ is the number of edges with which $v$ is incident. (Equivalently, $d(v)$ is the number of vertices to which $v$ is adjacent; i.e.,

$$d(v) = |\{u \in V(G) | \{u, v\} \in E(G)\}|.)$$

If the vertices of $G$ are labeled, $G$ is said to be labeled graph.

For brevity, we usually write $uv$ for $\{u, v\}$; $p = |V(G)|; q = |E(G)|$. The order of $G$ is given by $p$. The size of $G$ is given by $q$.

Example: Let $G$ be defined by:

$$V(G) = \{v_1, v_2, v_3, v_4\}$$
$$E(G) = \{v_1v_2, v_2v_3, v_3v_1, v_1v_4\}$$

then $G$ may be represented by either Figure 2-1a or 2-1b, where the latter representation is more accurate, in a sense we will describe in Chapter 6.